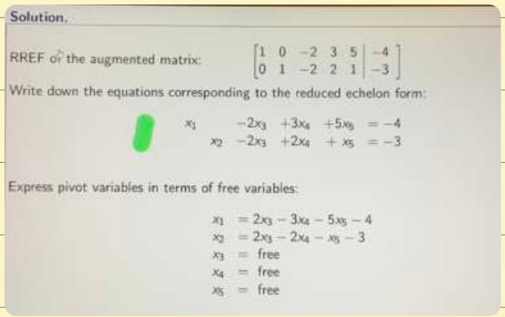
Gaussian Elimination

Solve linear systems for the pivot variables in term of free variable

1. write down argument matrix
2. Find Reduced echelon form
3. write down equations corresponded to reduced echelon form
4. Express pivot variables in terms of free variable

eg.


Solution.

RREF of the augmented matrix: $\left[\begin{array}{ccccc|c} 1 & 0 & -2 & 3 & 5 & -4 \\ 0 & 1 & -2 & 2 & 1 & -3 \end{array}\right]$

Write down the equations corresponding to the reduced echelon form:

$$\begin{array}{rrrrrl} x_1 & & -2x_3 & +3x_4 & +5x_5 & = -4 \\ & x_2 & -2x_3 & +2x_4 & +x_5 & = -3 \end{array}$$

Express pivot variables in terms of free variables:

$$\begin{aligned} x_1 &= 2x_3 - 3x_4 - 5x_5 - 4 \\ x_2 &= 2x_3 - 2x_4 - x_5 - 3 \\ x_3 &= \text{free} \\ x_4 &= \text{free} \\ x_5 &= \text{free} \end{aligned}$$

Free variable: Can be any val you like

Consistent: have solution

Theorem 1: A linear system is consistent if and only if an echelon form of the argument matrix has no row of the form $[0 \cdots 0 \mid b]$, where b is non-zero. if linear system consistent, then, the linear system has

1. A unique solution
2. infinitely many solution.

eg. $\left[\begin{array}{cc|c} 3 & 4 & -3 \\ 3 & 4 & -3 \\ 6 & 8 & 5 \end{array}\right] \rightarrow \begin{bmatrix} 3 & 4 & -3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

No solution because there is a $[0\ 0\,|\,0]$ in RREF

Linear Combination

Definition. Consider $m \times n$-matrices $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$, and $B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix}$.

a) The sum of $A + B$ is

$$\begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2n}+b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \cdots & a_{mn}+b_{mn} \end{bmatrix}$$

b) The product cA for a scalar c is

$$\begin{bmatrix} ca_{11} & ca_{12} & \cdots & ca_{1n} \\ ca_{21} & ca_{22} & \cdots & ca_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ ca_{m1} & ca_{m2} & \cdots & ca_{mn} \end{bmatrix}$$

eg. $\begin{bmatrix} 1 & 0 \\ 5 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 8 & 3 \end{bmatrix}$

$5\begin{bmatrix} 2 & 1 & 0 \\ 3 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 10 & 5 & 0 \\ 15 & 5 & -5 \end{bmatrix}$

- Addition is only define when A and B have same row & Colume

Colume Vector: $m \times 1$ - matrix

Row vector: $1 \times n$ - matrix

Transpose: if A is $m \times n$, the transpose of A is $n \times m$ matrix, denoted as A^T, whose $(A^T)_{ij} = A_{ji}$

eg. $\underset{A}{\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 0 \end{bmatrix}} \rightarrow \underset{A^T}{\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}}$

Linear Combination: linear combination of $m \times n$ matrix $A_1, A_2 \cdots A_p$, with coefficients $c_1, c_2 \cdots c_p$ is define as:

$$c_1A_1 + c_2A_2 + \cdots + c_pA_p$$

Span: Set of Linear Combination for $A_1, A_2 \cdots A_p$

$\mathbb{R}^m$: Set of all colume vector

eg. $a_1 = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$, $a_2 = \begin{bmatrix} 4 \\ 2 \\ 14 \end{bmatrix}$ and $b = \begin{bmatrix} -1 \\ 8 \\ -5 \end{bmatrix}$, is a a linear combination for a_1 and a_2?

$$x_1 \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 2 \\ 14 \end{bmatrix} = \begin{bmatrix} -1 \\ 8 \\ -5 \end{bmatrix}$$

$$\Rightarrow \begin{aligned} 1x_1 + 4x_2 &= -1 \\ 0x_1 + 2x_2 &= 8 \\ 3x_1 + 14x_2 &= -5 \end{aligned} \Rightarrow \left[\begin{array}{cc|c} 1 & 4 & -1 \\ 0 & 2 & 8 \\ 3 & 14 & -5 \end{array}\right]$$

$R_3 \to R_3 - 3R_1$

$$= \left[\begin{array}{cc|c} 1 & 4 & -1 \\ 0 & 2 & 8 \\ 0 & 2 & -2 \end{array}\right]$$

$R_3 \to R_3 - R_2$

$$= \left[\begin{array}{cc|c} 1 & 4 & -1 \\ 0 & 2 & 8 \\ 0 & 0 & -10 \end{array}\right]$$

inconsistent

$\Rightarrow b$ is not linear combination of a_1 and a_2

Matrix multiplication

Let x be a vector in $\mathbb{R}^n$ and $A = [a_1, a_2 \cdots a_n]$ an $m \times n$ matrix, we define the product Ax by:

$$Ax = x_1 a_1 + x_2 a_2 + \cdots + x_n a_n.$$

- Ax is only defined if the # of entries of x = # of columns of A.

eg. $A = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \end{bmatrix}$ $x = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, determ Ax and Bx

$$Ax = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 3 \end{bmatrix} = 2\begin{bmatrix} 2 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

$$Bx = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 6 \end{bmatrix} + \begin{bmatrix} 6 \\ 3 \\ 15 \end{bmatrix} = \begin{bmatrix} 8 \\ 3 \\ 21 \end{bmatrix}$$

eg2. Consider the following vector

$$x_1\begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_2\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

Find a 2×2 matrix A such that (x_1, x_2) is a solution to the above equation if and only if

$$A\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}.$$

$A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$ $Ax = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1\begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_2\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}.$

Theorem 1: let A be $[a_1 \cdots a_n]$ be a $m \cdot n$-matrix and b in $\mathbb{R}^m$.

Then, the following are equivalent

- $(x_1, x_2 \cdots x_n)$ is the solution for the vector equ'n $x_1 a_1 + x_2 a_2 + \cdots x_n a_n = b$

- $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ is a solution to the matrix equation $Ax = b$.

- $(x_1, x_2, \cdots x_n)$ is the solution of the system $[A \mid b]$.

Take matrix as a machine.

eg. $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.
$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$$x \to \boxed{A} \to Ax$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_1 \end{bmatrix}$$

eg. Let A be an $m \times n$ matrix and B be an $k \times l$ matrix

$$\xrightarrow{x} \boxed{A} \xrightarrow{Ax} \boxed{B} \xrightarrow{B(Ax)}$$

$Ax \in \mathbb{R}^m$

To make it arbitory, $m = l$

Matrix Goemetric:

$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$: reflection $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$: projection eg. Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ Is $A(Bx) = B(Ax)$

No, because porjection and reflection do not commute.

Matrix Multiplication

Let A be an $m \times n$ matrix and let $B = [b_1 \dots b_p]$ be an $n \times p$ - matrix, We define

$$AB = [Ab_1 \; Ab_2 \cdots Ab_p]$$

eg. $A = \begin{bmatrix} 4 & -2 \\ 3 & -5 \\ 0 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 2 & -3 \\ 6 & -7 \end{bmatrix}$, $A \cdot B$?

$$Ab_1 = \begin{bmatrix} 4 & -2 \\ 3 & -5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 6 \end{bmatrix} = \begin{bmatrix} -4 \\ -24 \\ 6 \end{bmatrix}$$

$$Ab_2 = \begin{bmatrix} 4 & -2 \\ 3 & -5 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} -3 \\ -7 \end{bmatrix} = \begin{bmatrix} 6 \\ 26 \\ -7 \end{bmatrix}$$

$$\Rightarrow AB = \begin{bmatrix} -4 & 6 \\ -24 & 26 \\ 6 & -7 \end{bmatrix}$$

- Ab_1 is a linear combination of the columns of A from the corresponding colums of B

eg. C is 4×3 and D is 3×2. are CD and DC define? what is their size?

CD is define. 4×2. DC not define

Size: • The product of A and B only define if B has many rows as A has columns. In this case then AB has as many rows as A and as many colume as B.

• Let B be $n \times p$: input $x \in \mathbb{R}^p$, output $c = Bx \in \mathbb{R}^n$.

eg. $A = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

Compute $(AB)x$ and $A(B(x))$, Are they the same.

$A(B(x))$ $\quad B(x) = \begin{bmatrix} x_1 + 2x_2 \\ x_2 \end{bmatrix}$. $A(B(x))$

$= \begin{bmatrix} 2x_1 + 4x_2 \\ x_1 + 3x_2 \end{bmatrix}$

$(AB)x = \begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 + 4x_2 \\ x_1 + 3x_2 \end{bmatrix}$

Theorm: Let A be an $m \times n$ matrix and B be an $n \times p$ matrix,
Then for every $x \in \mathbb{R}^p$, we have:

$$A(Bx) = (AB)x$$

Row-Column Rule

Let A be the $m \times n$ and B be $n \times p$ such that

$$A = \begin{bmatrix} R_1 \\ \vdots \\ R_m \end{bmatrix}, \text{ and } B = [C_1 \cdots C_p].$$

Then: $AB = \begin{bmatrix} R_1C_1 & \cdots & R_1C_p \\ R_2C_1 & \cdots & R_2C_p \\ R_mC_1 & \cdots & R_mC_p \end{bmatrix}$ and $(AB)_{ij} = R_iC_j = a_{i1}b_{1j} + a_{i2}b_{2j}$

$+ \cdots a_{in} b_{nj}$

$$AB = C_1R_1 + \cdots + C_nR_n$$

eg. $A = \begin{bmatrix} 2 & 3 & 6 \\ -1 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -2 & -3 \\ 0 & 1 \\ 4 & -7 \end{bmatrix}$ Compute AB if define.

$$AB = \begin{bmatrix} 4+0+24 & 2(-3)+3\cdot 1+6\cdot(-7) \\ (-1)(-2)+0+4 & 3+0-7 \end{bmatrix} = \begin{bmatrix} 28 & -45 \\ 2 & -4 \end{bmatrix}$$