Defination

Linear Equation: $a_1x_1 + \cdots + a_nx_n = b$

where $a_1, a_2 \cdots$ are numbers and $x_1 - x_n$ are variables

eg. $4x_1 - 5x_2 + 2 = x_1 \iff 3x_1 - 5x_2 = -2$ ✓

$x_2 = 2(\sqrt{6} - x_1) + x_3 \iff 2x_1 + x_2 - x_3 = 2\sqrt{3}$ ✓

$4x_1 - 6x_2 = x_1x_2$ ✗

$x_2 = 2\sqrt{x_1} - 7$ ✗

Linear System: Collection of one or more linear equations

Solution: linear system's solution is a list $(s_1, s_2, s_3 \cdots s_n)$

eg.

(I) $x_1 + x_2 = 1$

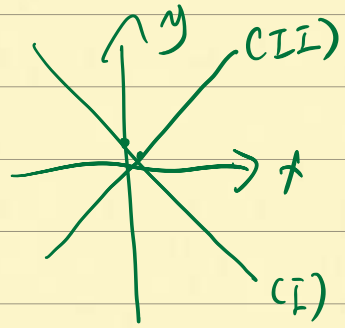
(II) $-x_1 + x_2 = 0$

Solution?

① Add I and II: $2x_2 = 1 \Rightarrow x_2 = 1/2$

② plug into I $x_1 + 1/2 = 1 \Rightarrow x_1 = 1/2$

③ Thus $(x_1, x_2) = (1/2, 1/2)$ is the only solution.



Not every system have only one solution.

eg.

$$x_1 - 2x_2 = 3 \quad (III)$$
$$2x_1 - 4x_2 = 8 \quad (IV)$$

① Multiply I by 2 $\Rightarrow 2x_1 - 4x_2 = -6$

② Substract (I) from (2V)

$\Rightarrow 0 = 14$ which is always falso.

$\Rightarrow$ No solution exist

eg.

$$x_1 + x_2 = 3 \quad (V)$$
$$-2x_1 - 2x_2 = 6 \quad (VI)$$

Multiply V by 2 $\Rightarrow 2x_1 + 2x_2 = 6$

V and VI have same value just set

$x_2 := 3 - x_1$ equation has inf solution

Summary: A linear system has either: $\begin{cases} \text{One unique solution} \\ \text{No solution} \\ \text{inf many solution} \end{cases}$

equivalent: Two linear systems are equivalent if they have the same solution set

Matrix and linear System

Defination: A $m \times n$ matrix is a rectangular array of number with m rows and n columns.

eg. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \end{bmatrix}$, $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$, $\begin{bmatrix} 1+\sqrt{5} \end{bmatrix}$

Entries : $A = \begin{bmatrix} a_{11} & a_{12} & \\ a_{21} & a_{22} & \cdots \\ \vdots & \vdots & \\ a_{m1} & a_{m2} & \end{bmatrix}$ a_{ij} : i row j colume

Definition. For a linear system, we define the **coefficient** and **augmented** matrix as follows:

linear system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

coefficient matrix

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

augmented matrix

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

Elementary row operation

① Replacement : Add a multiple of one row to another row.

$R_i \to R_i + cR_j$, where $i \neq j$

② Interchange : Interchange two rows : $R_i \Leftrightarrow R_j$

③ Scaling : Multiply all entries in a row by a nonzero constant : $R_i \to cR_i$, where $c \neq 0$

eg. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow[\text{Replacement}]{R_2 \to R_2 + 3R_1} \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 3 \end{bmatrix} \xrightarrow[\text{Interchange}]{R_1 \Leftrightarrow R_3} \begin{bmatrix} 2 & 3 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{R_2 \to 3R_2} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$

eg. Consider a row operation $R_3 \to R_3 + 3R_1$, Is there any operation reverse it?

$$R_3 \to R_3 - 3R_1$$

Replacing

Summary: Every Row operation is reversible, like:

Scaling : $R_2 \to cR_2$: $R_2 \to \frac{1}{c} R_2$.

Interchange : $R_1 \leftrightarrow R_2$: $R_1 \leftrightarrow R_2$

Row equivalent: Two matrix are row equivalent if one matrix can be transform into the other matrix by a sequence of elementary row operation

Theorem 1: If the argumented matrix of two linear systems are row equivalent, then the two systems have the same solution set

eg.
$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_3 \to R_3 + 3R_1]{} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}}_{\text{Row Equivalent}} \xrightarrow[R_3 \to R_3 - 3R_1]{} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Echelon form (or row echelon form)

Define: A matrix is in Echelon form if:

① All non zero rows (rows with at least one nonzero element) are above any rows of all zeros.

② The leading entry (the first nonzero number from left) of a nonzero row is always strictly to the right of the leading entry of the row above it.

eg. $\begin{bmatrix} 3 & 1 & 2 & 0 & 5 \\ 0 & 2 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ ← nonzero row 1. ✓

Leading Entry $3 \searrow 2$ Right 2. ✓

⇒ True

$\begin{bmatrix} 0 & 2 & 0 & 1 & 4 \\ 3 & 1 & 2 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ 1. ✓ 2. ✗ ⇒ False

$\begin{bmatrix} 2 & -2 & 3 \\ 0 & 5 & 0 \\ 0 & 0 & 5/2 \end{bmatrix}$ 1. ✓ 2. ✓ ⇒ True

$\begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{bmatrix}$ 1. ✓ 2. ✓ ⇒ True

Row Reduced echelon Form (or reduced echelon form or RREF)

Requir: ① is echelon form

② The leading entry in each non-zero row is 1

③ Each leading Entry is the only non-zero entry in the column

eg.

a) $\begin{bmatrix} 0 & 1 & 3 & 0 & 0 & 2 & 5 & 0 & 0 & 6 \\ 0 & 0 & 0 & 1 & 0 & 1 & \frac{1}{2} & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 1 & -3 & 4 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$

① ✓
② ✓ ⇒ True
③ ✓

$$\begin{bmatrix} 1 & 0 & -2 & 3 & 2 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

X

① ✓
② ✓ = False
③ X

Theorem 1 Each matrix is row equivalent to one and only one matrix in reduced echelon form

def: We say that matrix B is RREF of matrix A if A and B are row-equivalent and B is in RREF.

eg. Find RREF of the matrix:

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \end{bmatrix}$$

① $R_2 = R_2 - R_1 \Rightarrow \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \end{bmatrix}$

② $R_1 = R_1 \cdot 1/3$

$R_2 = R_2 \cdot 1/2 \Rightarrow \begin{bmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 1 & -2 & 2 & 1 & -3 \end{bmatrix}$

③ $R_1 = R_1 + 3R_2 \Rightarrow \begin{bmatrix} 1 & 0 & -2 & 3 & 5 & -4 \\ 0 & 1 & -2 & 2 & 1 & -3 \end{bmatrix}$

Pivot position: the position of the leading entry in an echelon form of a matrix. the pivot column is the column that contains a pivot position

eg.

Example. Locate the pivot columns of the following matrix.

$$A = \begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$$

Solution.

$$\underset{R_1 \leftrightarrow R_3}{\rightsquigarrow} \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix} \underset{R_2 \to R_2 + R_1}{\rightsquigarrow} \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

$$\underset{R_3 \to R_3 + 1.5R_2}{\rightsquigarrow} \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix}$$

Thus columns 1, 2, 4 of A are the pivot columns of A.

Basic variable (or pivot variable): is a variable that corresponds to a pivot column in the coefficient matrix of a linear system. A **free variable** that is not a pivot variable.

eg.

$$\begin{array}{ccccc|c} & & & & & \\ \end{array}$$
$$\left[\begin{array}{ccccc|c} 1 & 6 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & -8 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{array}\right]$$
$$\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array}$$

$\Rightarrow$ x_1, x_3, x_5 are basic variable

x_2, x_4 are free variable